SCIRES it
SCIentific RESearch and Information Technology
Ricerca Scientifica e Tecnologie dell'Informazione
Vol 15, Issue 1 (2025), 141--158
e-ISSN 2239-4303, DOI 10.2423/i22394303v15n1p141

CASPUR-CIBER Publishing, http://www.sciresit.it

Geometry, Pedagogy, and Printing Techniques: An Analysis of Euclid's Dodecahedron

Massimiliano Ciammaichella, Chiara Monaco**, Luciano Perondi**

*Università Iuav di Venezia – Venezia, Italy.
**Independent Researcher – Italy.

Abstract

The article examines visual representations of Euclid's dodecahedron (Book XIII, Proposition XVII), emphasizing Erhard Ratdolt's influential 1482 edition, notable for innovative diagrams and clear pedagogical intent. By tracing graphic evolution across key editions, including those by Tartaglia (1560) and Commandino (1572) it explores gradual advances in depicting geometric forms, such as visual hierarchy through varied line thicknesses, color differentiation, and perspective-based constructions. However, widespread adoption of these innovations was significantly delayed by technical constraints of early printing methods, adherence to traditional visual standards, and cautious experimentation. The analysis thus highlights a continuous dialogue between mathematical abstraction, graphic representation, and pedagogical needs.

Keywords

Dodecahedron, Graphic Phylogeny, Ratdolt

1. *Prefiguring and representing Euclidean dodecahedra*

The present collection of various translations of *Euclid's Elements* focuses on representations of the dodecahedron, spanning a period from approximately 888 CE—with the Greek manuscript edition (MS. D'Orville 301)—to the *Opera Omnia* edited by Heinrich Menge and Johan Ludvig Heiberg (1885).

Sixty volumes document the millennia-long recurrence of the generative processes associated with the Platonic solid[1], oscillating between geometric genesis and empirical practice. These works demonstrate its principles through engravings that, while illustrative, often complicate its interpretation.

Starting from the enunciative synthesis of a well-known problem—presented in Proposition 17 of Book XIII, written by the renowned Greek mathematician and philosopher—the objective is to construct a polyhedron with twelve pentagonal faces inscribed in a sphere. This construction demonstrates that its edges are portions of irrational straight lines, defined in terms of apotomes[2] (Knorr, 1989), since they are derived from the golden ratio in relation to the edges of a cube, which is also inscribed in the same sphere. The vertices of this cube coincide with eight of the twenty vertices of the dodecahedron to be represented.

In fact, the translation of the proposed problem into a drawing can be summarized and simplified by tracing the pentagonal profile, whose angulation is oriented around the edge shared by two faces of the cube, which coincides to one of the five diagonals of the pentagon in question.

One of the possible demonstrations, validated by the fundamentals of descriptive geometry, can be traced back to the representation in isometric axonometry (fig. 1) of a cube, of which we consider the two adjacent faces ABCD and ABEF, together with the midpoints of the segments that divide them into four square modules, in the centers K and X respectively. A further subdivision is made by the segment NO intersecting GH at point P, considered as the center of a circumference of radius PJ intersecting GH at point Q; then a second circumference with center K and radius KG can be

[1] For further details, see the timeline by Autors: link.

[2] Book XIII, Proposition 6: *If a rational straight line is cut in extreme and mean ratio, then each of the resulting segments is the irrational straight line called an apotome.*

drawn, identifying the points M and L lying on the segment IJ. Euclid states that these points divide the segments KI and KJ into extreme and mean ratio, because the relationship between IK and IM, KG and QG, KJ and LJ coincide with the golden section. In particular, IM is equal to MR, LS and GV. This allows us to trace the pentagonal face AUBSR and thus the entire dodecahedron.

A key aspect of this subject is the manner in which historical treatises have visually represented the generative and constructive processes of the Platonic solid—a symbol of the universe. This concept, already familiar to the classical world, was attributed to the philosopher Philolaus, who, in the 5th century BCE, disseminated Pythagorean doctrine and recognized its origin (Joost-Gaugier, 2008). However, a comparison between the drawings in the first editions of the D'Orville manuscript and the version preserved in the Vatican Apostolic Library (Vat. gr. 190, pt. 2) reveals that the reference to the orthogonality of the two cube faces suggests an early intuition of Cavalier axonometry. This intuition appears accurate only in demonstrations where non-oblique surfaces are correctly represented with respect to the picture plane. This is exemplified by the pentagonal profile, in which the projective conditions of parallelism and orthogonality are not satisfied, resulting in a representation that is difficult to interpret.

Although terms such as 'to raise' may evoke Mongean methods, the use of wireframe representation abstracts the objects from their solidity, emphasizing the mathematical definitions articulated by Euclid in the first book. According to these definitions, a surface consists only of length and width, a line consists only of length with endpoints as points, and a point is 'that which has no parts', making it, by its very nature, impossible to represent.

The metaphysical and non-corporeal conception of objects seems to rely on the diagram, because it "has no dimensions and is not visible in any real space, i.e. it is not drawn from it. For this reason, Aristotle states that only diagrams and schematics are acceptable for illustrating scientific texts, and sometimes they almost take the place of the theorem itself" (Scolari, 2005, p. 213). This principle is consistently upheld in other translations, which prioritize the graphic demonstration of a mathematical concept over its portrayable appearance. As a result, the austere compositional aesthetic remains unchanged, with exceptions in the construction process. These exceptions include inversions of the two cube faces and distortions of proportions that bring them closer to coplanarity. Such deviations are evident in the analysis of two fifteenth-century manuscripts translated by the astronomer Campano da Novara (Urb. Lat. 507; Urb. Lat. 506).

In Piero della Francesca's experience, on the other hand, mathematical abstraction is confronted with concreteness in the understanding of the rules of plane and solid geometry, invoking the dodecahedron to depict it in all its physicality, both in the Trattato d'abaco (1460-1480) and in De quinque corporibus regularibus (Urb. lat. 632), written between 1476 and 1500. This drew attention to how, for the artist, "the deductive perfections of mathematics provide an active, a priori, model for our understanding of experience, rather than arising simply from an empirical study of the sensory world" (Kemp, 1990, p. 27).

His work greatly influenced that of Luca Pacioli, who devoted the third part of his treatise *Divina proportione* (1509) to translating *De quinque corporibus regularibus* from Latin into the vernacular, without citing the author. Perhaps he thought that the only copy he had studied years before had been lost (Fazzini, 2003); in fact, Giorgio Vasari accused "Maestro Luca da'l Borgo friar of Saint Francis" (1550, p. 365) of plagiarism. The possibility that he was a disciple of Piero della Francesca is supported by historical records indicating the presence of both individuals in San Sepolcro during the second half of the 1480s. However, the inherently 'practical' nature of Euclidean geometry is intertwined with Platonic philosophical themes, particularly those related to the five elements described in the *Timaeus*, albeit with certain textual modifications (Ciocci, 2009).

From a general perspective, the iconographic apparatus operates on two distinct stylistic registers. On one hand, the pictorial and figurative aspect is conveyed through the perspectival artifice of full-page drawings—exemplified by Leonardo da Vinci's depiction of the dodecahedron, both solid and hollow, with a pentagonal face parallel to the picture plane. On the other hand, the diagrammatic construction of the polyhedron adheres to the established principles of earlier treatises, appearing as a smaller illustration accompanying the text. In a similar vein, Albrecht Dürer engaged in practical demonstrations of Euclidean postulates following his journey to Italy, where he acquired a copy of Elements, translated from Greek by Bartolomeo Zamberti (1505).

The fourth and last part of his treatise *Underweysung der Messung* (1525) is devoted to the study of polyhedra, and about the dodecahedron, following Leonardo's example, he obtains the templates of its development in order to reconstruct it in maquette, after having skillfully represented it in plan and elevation.

With a few exceptions, the translations of the sixteenth and seventeenth centuries seem to have favored the already familiar diagrammatic scheme, in pseudo-oblique axonometries and with one side parallel to the plane of representation. problems, by privileging the rules of central perspective in a praise of the depth of space. But it is also a critique of the complexity of philological images produced by copyists, in Greek and Latin translations. Thus Bernardino Baldi, student and biographer of Federico Commandino, states: "In addition to the sincerity of the language, he is remarkable for the diligence of the figures, in which, having employed the art of perspective, he disgusted at those ugliness in which incur and incur those who went after the depraved tradition to the barbarous custom; and well can I note this fact, because being a young man myself, and attending with much gentleness to these studies, I drew with much patience a great number of them" (Nenci, 1998, p. 518).

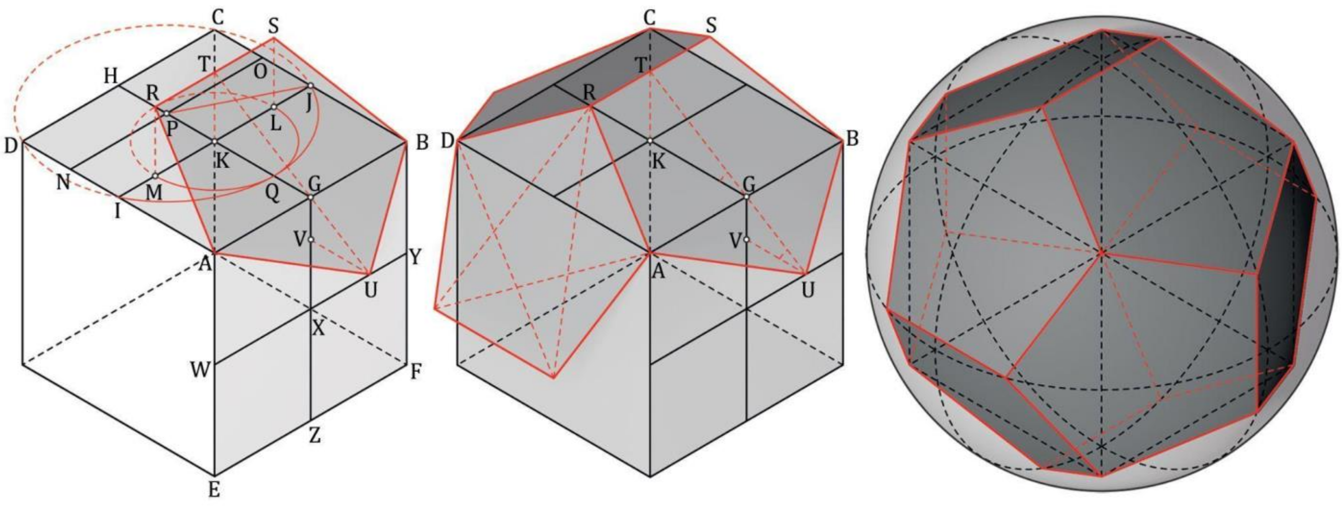

Fig. 1: Construction of the pentagonal face and dodecahedron inscribed in a sphere. Isometric axonometries.

However, one might be surprised by the use of central perspective proposed by Federico Commandino (1572). He inverts the two faces of the cube by considering its inner edge, which is used to orient the construction of the pentagon.

In fact, it is well known how the sixteenth century consolidated the Italian scientific codification of central projection at the expense of parallel projection. However, mathematicians of the caliber of Commandino, Ignazio Danti and Guidobaldo Del Monte contributed to the birth of projective geometry, to be understood in the terms of an autonomous science. Then, the precise choice to translate the iconographic apparatus of the Elements into subjective rather than objective projections explain the desire to articulate constructive processes in demonstrations of

2. *Note on the Numbering of Books and Documents Containing the Representations*

We have developed a numbering system for this article. We numbered the representations of the dodecahedron found in the various editions of Euclid's Elements, as no pre-existing numbering was identified in the literature. This numbering follows the chronological order of the date of creation or publication (in the case of printed works). The numbering increments by 10 to allow for the potential inclusion of additional works that may currently be unknown to us.

We have separately numbered the figures that we identified as being in some way connected to The Elements by Euclid, using the same system but prefixing the work's number with 'NE'.

We elaborated an hypothesis of timeline and phylogenesis (fig. 2 and fig. 3) of the translation and of the images in order to understand the evolution of the dodecahedron representation.

Our analysis is focused on the evolution of the dodecahedron representations between 15th and 16th century, so we included the known sources and some of the leather edition of the 17th and 18th century in order to display the evolution of the representation.

3. *A brief phylogeny of Translations*

The iconographic trajectory of the Euclidean dodecahedron demonstrates an interplay between theoretical abstraction and graphic practice, reflecting shifts in mathematical knowledge transmission. This balance is mirrored in the philological history of Euclid's Elements translations. The textual evolution conveys the geometric and philosophical nature of the work.

Euclid's Elements, dating around 300 BCE, is the most significant mathematical treatise preserved from Greek antiquity, although Euclid's original text is lost. The earliest version is by Theon of Alexandria (4th century), who mistakenly attributed the last two books (14th and 15th) to Euclid; these likely belong to Hypsicles of Alexandria (Riccardi, 1887, p. 6 (404)).

Translations of Elements split into Greek and Arabic traditions until Stapulensis's 1516 edition. Arabic translators often conveyed meaning rather than literal translations, complicating direct connections to the Greek text (De Young, 2004). Translation activities flourished in the 12th century in two streams: from Arabic, primarily in Spain, and from Greek, mainly in Sicily and Constantinople (Pergola, 2009).

The main Latin translations from Arabic are by Adelard of Bath, producing three versions: Adelard I, the earliest known (Pergola, 2009); Adelard II, incorporating multiple sources, likely compiled by Robert of Chester and later used by Campanus of Novara (Pergola, 2009); Adelard III, a commentary attributed to John of Tynemouth (Pergola, 2009).

Campanus of Novara's Elementa in artem geometriae et Campani commentationes (c. 1255) is more commentary than translation, based on Adelard II and a Greek text from Norman Sicily (Bernante 2020, Gavagna 2010). Campanus' version shaped two major translation branches and influenced manuscripts such as Vat. lat. 2224, commissioned by Francesco Cereo da Borgo Sansepolcro (Ciocci, 2020).

Campanus' edition gained prominence through Ratdolt's editio princeps (Venice, 1482), notable for its innovative diagrams, which are discussed below. Ratdolt studied under the astronomer Johann Müller (Regiomontanus), whose scholarly program influenced Ratdolt's choice of Campanus's text (Bernante, 2020).

Contrasting Campanus, Bartolomeo Zamberti translated Elements directly from Greek, aligning with Humanist ideals. Published by Giovanni Tacuino in 1505, Zamberti criticized Campanus for deviating from Greek originality (Gavagna, 2010; Bernante, 2020). Campanus emphasized mathematical coherence, while Zamberti prioritized linguistic fidelity to the Greek source (Bernante, 2020).

4. *Phylogeny of the representations. Evolution in the construction of picture hierarchies towards Ratdolt edition*

The complexity inherent in the textual transmission of Euclid's Elements is equally reflected in the phylogeny of its graphic representations. Examining the drawings introduced in the first chapter demonstrates a gradual evolution in representation, enhancing ease of identification and interpretation of geometric elements (fig. 4 and fig. 5).

This progressive refinement suggests an increasing awareness of the strategic use of visual elements to distinguish graphical components. Nonetheless, a distinct hierarchical differentiation based on line thickness emerged in the press notably late, appearing explicitly only in the Elements published in the 19th century. This gradual development likely involved an implicit acquisition of communicative strategies aimed at semantically structuring elements within illustration.

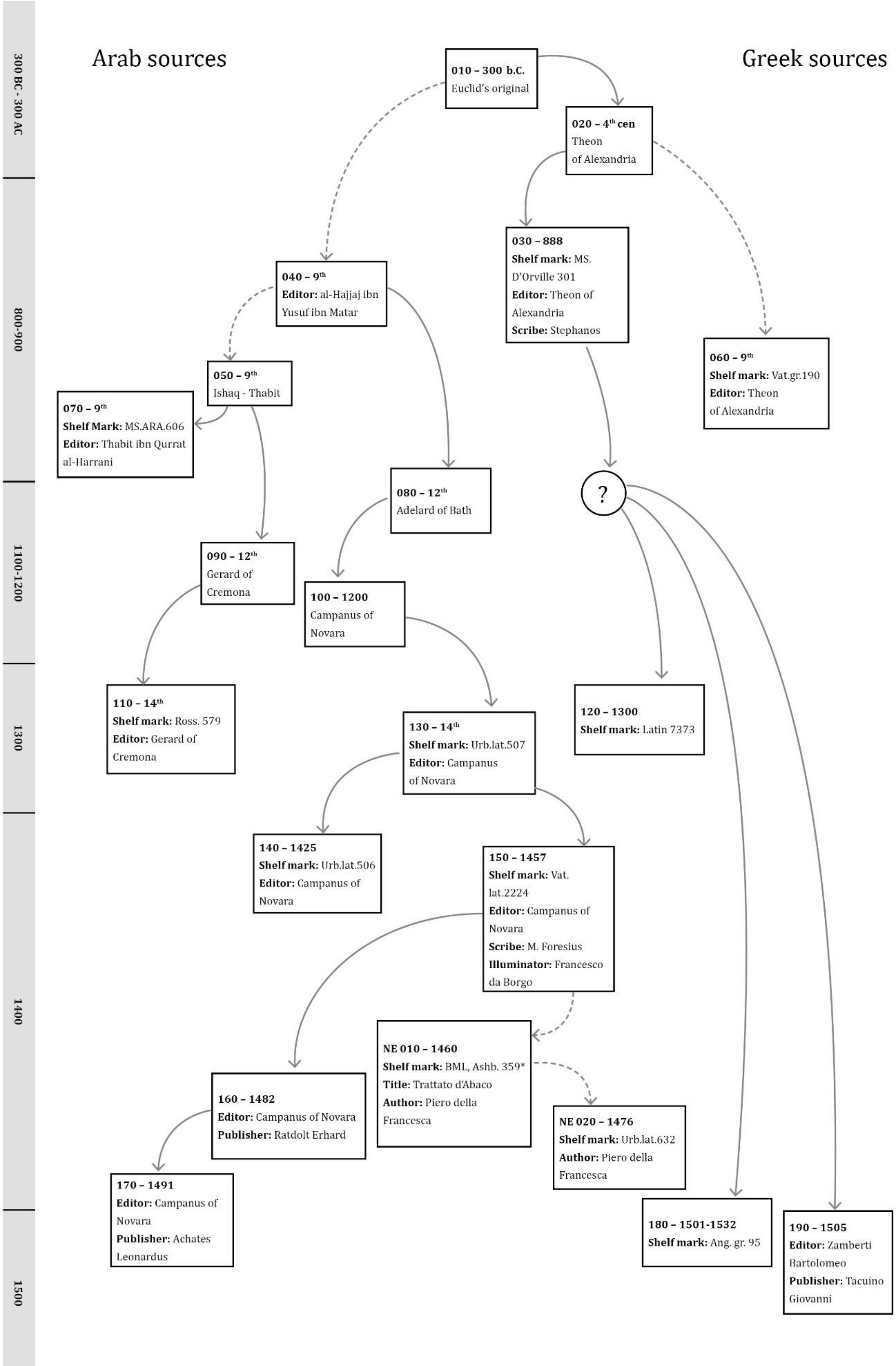

Fig. 2: phylogeny of Euclid's *Elements* translations (1)

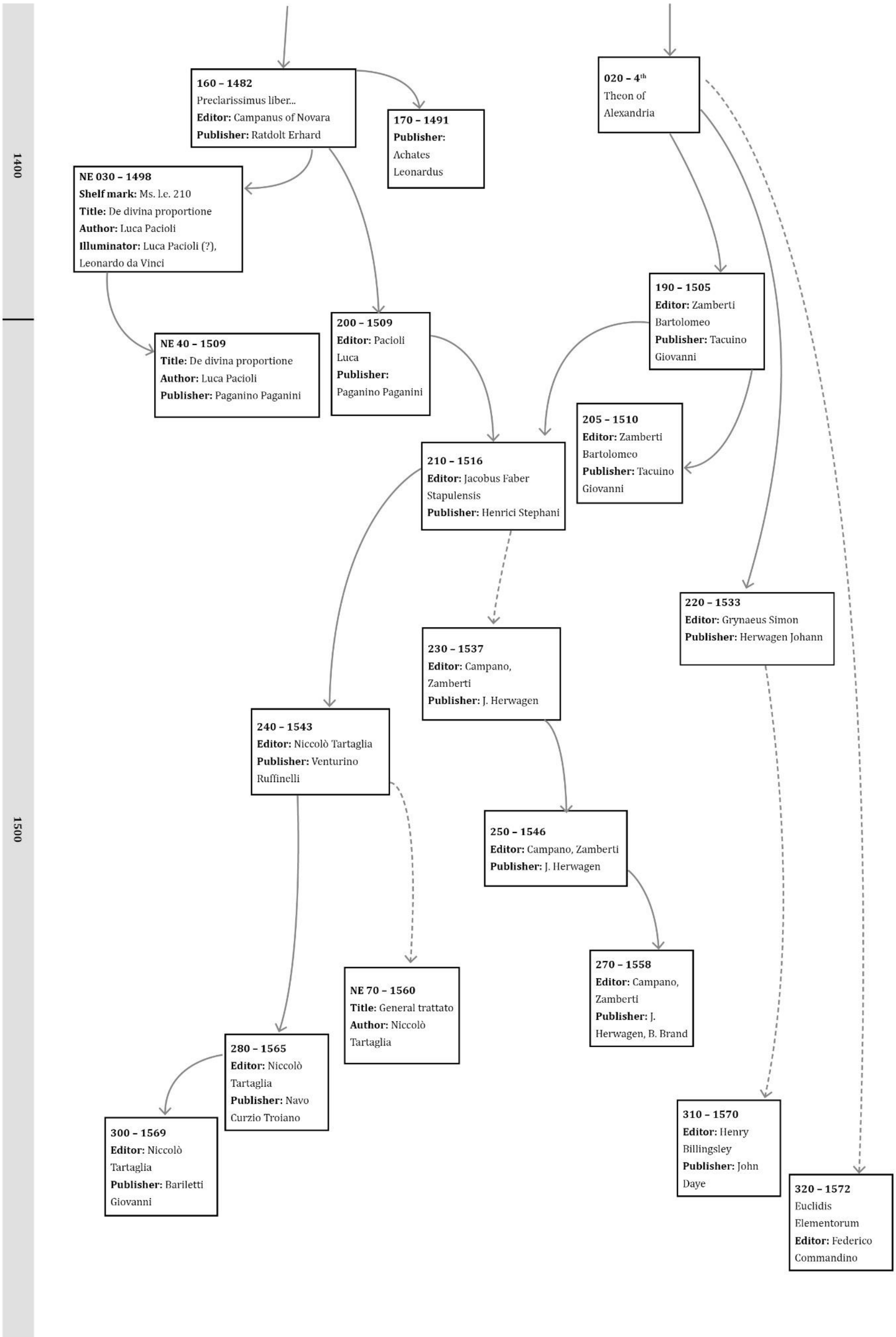

Fig. 3: phylogeny of Euclid's *Elements* translations (2)

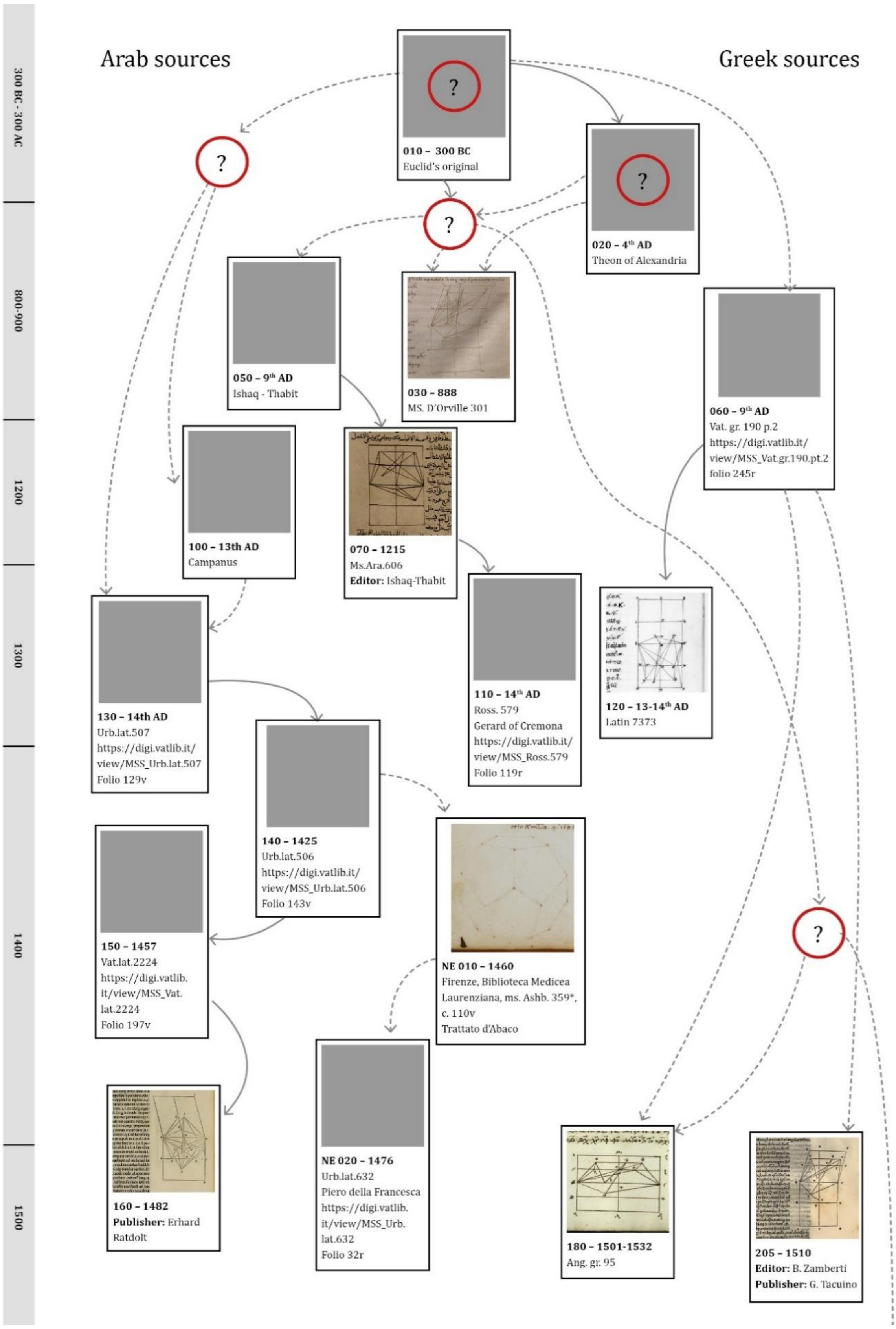

Fig. 4: Phylogeny of the dodecahedron in Euclid's Elements (1). Trattato d'Abaco (ms. Ashb. 359*, c. 110v). Held at the Biblioteca Medicea Laurenziana, Florence. Reproduced by the authors with permission of the Italian Ministry of Culture (MiC). All further reproduction by any means is prohibited. 030 – MS. D'Orville 301. CC BY-NC 4.0. 120 – Latin 7373 Source gallica.bnf.fr / BnF. All the other images are under Public Domain Mark 1.0.

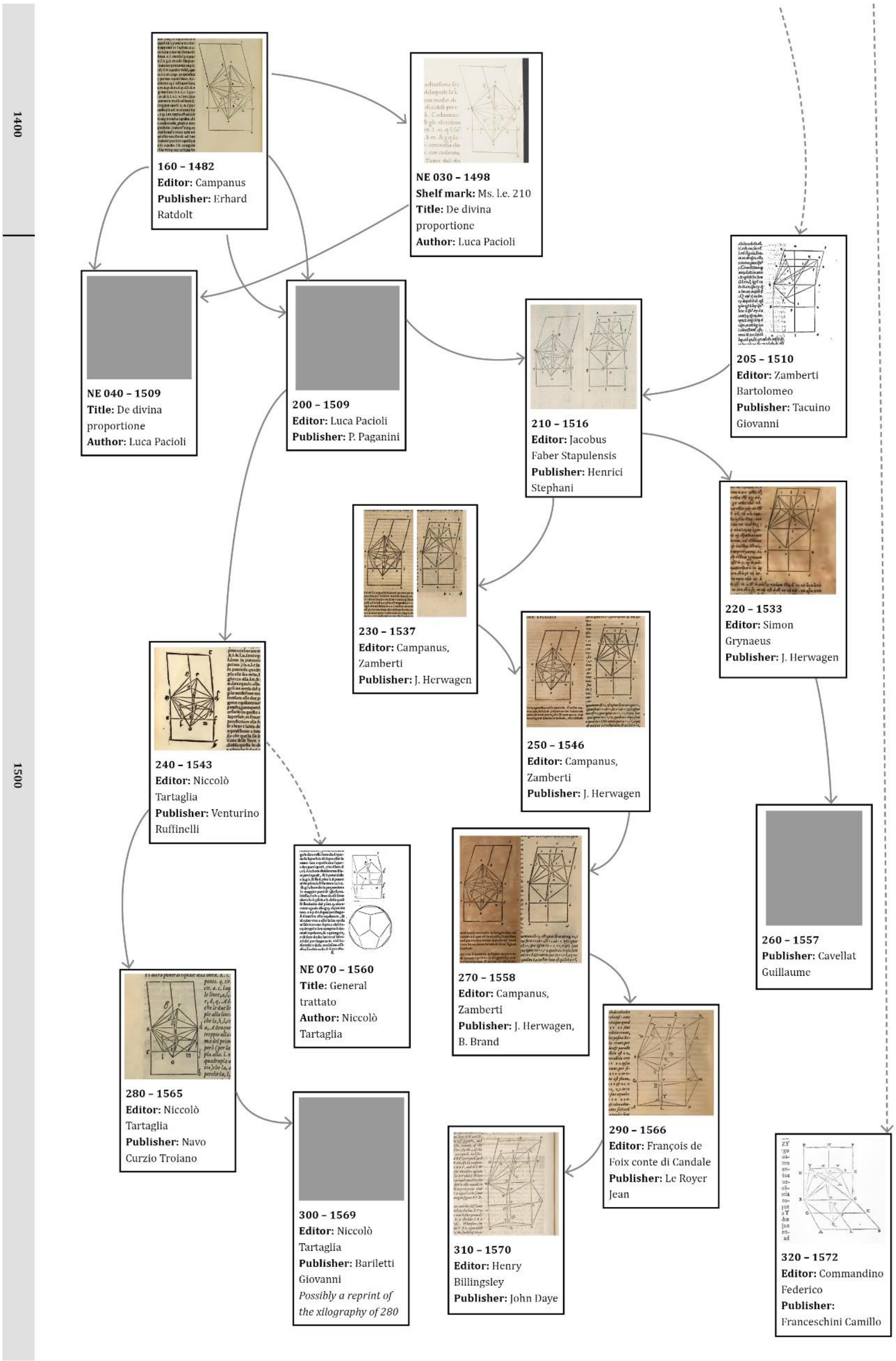

Fig. 5: Phylogeny of the dodecahedron in Euclid's Elements (2). NE 030 – Ms. l.e. 210. CC BY-NC 4.0. All the other images are under Public Domain Mark 1.0. 180 – Ang. gr. 95 CC BY-NC-SA. 310 – QA31 .E87 Fair use. All the other images are under Public Domain Mark 1.0.

4.1 Visual Hierarchy and Line Thickness

Prior to the 19th century, printed illustrations notably lack a hierarchical definition through variations in line thickness. Even Ratdolt's highly accurate and detailed 1482 edition maintains uniform line thickness, precluding the establishment of clear visual hierarchies. Conversely, hand-drawn illustrations, such as those found in the *Elementa* by Campano da Novara (Vat.lat.2224 (150)), created by Francesco Del Borgo in 1457[3], employed color differences to achieve visual differentiation. Baldasso (2008) emphasizes the precision of Ratdolt's linework compared to contemporary and later editions—a standard matched only much later by the 1566 edition of *Elementa* by Jean Le Royer, which itself was based on Francesco Cereo's earlier edition of Campano da Novara (Ciocci, 2020). Within these editions of *Elements,* the lack of hierarchical visual cues often compromises clarity, notably causing difficulty in distinguishing geometric features, such as pentagons or cube faces, from standard line segments.

4.2 Ratdolt's Pedagogical Vision

Early representations related to Book XIII, Proposition XVII of *The Elements* illustrate two seemingly distinct groups. The first group includes the oldest surviving examples, such as Vat. gr. 190 (060) and MS. D'Orville 301 (030), later utilized in Tacuino's printed edition (*Euclide, Elementa Geometriae*, Venezia: Tacuino, 1510–205). The second group centers around the image from Urb. Lat. 507 (040), closely resembling Ratdolt's 1482 edition (*Euclide, Elementa Geometriae*, Venezia: Ratdolt, 1482–160). These parallel evolutionary paths indicate that printed figures closely mirrored their hand-drawn counterparts, demonstrating a direct lineage despite minor variations such as mirrored or slightly altered configurations (fig. 10).

Although Ratdolt's groundbreaking figures were fundamental for disseminating mathematical and scientific knowledge, implementing a visual hierarchy through line variation was likely challenging or not convenient even for an innovative printer like him, despite his known capability for multicolor printing. Ratdolt was explicitly aware of the pedagogical importance of images, recognizing the intricate relationship between geometric figures and mathematical comprehension. He explicitly addresses this concern in the preface to his 1482 edition: "When I discussed these matters with myself more frequently, I found that this occurred due to the difficulty of the work [of studying mathematics]. Indeed, they had not yet figured out how to represent geometric figures, which are latent in mathematical volumes and without which almost nothing in these disciplines can be well understood. [...] Therefore, I hope that with this invention of ours, these disciplines that the Greeks call mathematics will soon be illuminated by a great number of volumes, just like the other sciences"[4] (Ratdolt, 1482, as translated by the author).

Baldasso (2008) further observes that Ratdolt's single, highly condensed figures frequently represented concepts that typically required multiple sequential illustrations, an approach also adopted later in Byrne's 1847 edition—although Byrne's coverage was limited to the first six books, excluding the dodecahedron. Throughout the sixteenth century, illustrations accompanying *The Elements* gradually evolved to serve explicitly didactic purposes (Lee, 2018). These figures increasingly illustrated step-by-step constructions, incorporating visual cues such as compass marks, which were already evident in Ratdolt's edition and further exemplified in Vögelin's 1528 printing.

4.3 Slow Adoption of Visual Innovations

Regarding the dodecahedron specifically, illustrations remained largely unchanged until Tartaglia's comprehensive depiction in *La quinta*

[3] Vat.lat.2224 seems to be the nearest antecessor of Ratdolt picture, it shares notable similarities also with figures from De quinque corporibus regularibus by Piero della Francesca (Urb.lat.632–NE 020). The relationship between these figures is plausible, given that Francesco del Borgo and Piero della Francesca were contemporaries and likely relatives (Banker, 2003). Further support arises from Piero's documented reliance on Francesco's commissioned copy of The Elements, uniquely identified by its distinctive numbering of certain Archimedean propositions (Pagliara, 1997).

[4] Hec cum mecum saepius discuterem inveniebam id difficultate operis accidisse. Non enim adhuc quo pacto schemata geometrica, quibus mathematica volumina latent, ac sine quibus nihil in his disciplinis fere intelligi optime potest excogitaverant [sic]. [...] Quam ob rem, ut spero, hoc nostro invento he discipline [sic] quas mathemata greci [sic] appellant voluminum copia sicuti reli(n)que scientie [sic] brevi illustrabuntur.

parte del general trattato (1560). Subsequent editions, particularly Le Royer's 1566 version edited by François de Foix de Candale, introduced significant technical enhancements, refining the depiction of complex geometric forms.

Despite Ratdolt's clear educational intent and the varying quality of illustrations throughout editions, substantial didactic advancements in graphical representation were sparse until the 19th century, when significant shifts in the use of visual variables occurred. Although seemingly straightforward, the employment of varied line thicknesses to establish visual hierarchies proved complex and non-intuitive. Indeed, earlier solutions, such as Campano da Novara's use of dual colors instead of differing line thickness, illustrate that effective visual strategies emerged slowly, refined gradually through experimentation and adaptation.

This slow progression is further underscored by editions frequently replicating original illustrations without significant innovation. An illustrative example is Commandino's perspective-based solution, which was not revisited until Flaminio Ingegneri's 1619 edition. This delay may reflect limited resources, reduced emphasis on didactical functionality, or hesitation to diverge from established traditional imagery.

5. *Ratdolt's Technique*

The presence of diagrams in Euclid's *Elements* is crucial for understanding the mathematical text. However, with the invention of the printing press, it became particularly challenging to print figures alongside the text. In most cases, these diagrams were produced using woodcut techniques, which involved carving a wooden block to remove the parts that were not part of the image. Woodcuts were particularly suitable for printing with movable type since both required the same type of press due to their raised surfaces.

Among the various editions of the *Elements*, the most technically intriguing is likely the one printed by Erhard Ratdolt in 1482 in Venice, with the text by Campano da Novara. As noted by Baldasso (2008), it is unlikely that the diagrams in this edition were woodcuts: the lines maintain a consistent thickness, even at intersections, and no wood grain is visible, which is typical of woodcuts. However, Stephan Füssel (1999, in Baldasso, 2008) asserts that these diagrams were indeed woodcuts.

Ratdolt himself hints at a new technique in the dedicatory letter he wrote to Doge Giovanni Mocenigo, found at the beginning of the Elements edition. In this letter, Ratdolt emphasizes the importance of diagrams in mathematical texts and refers to a new technique he invented for printing these figures, although he does not specify the exact method (Baldasso, 2008). Various hypotheses have been proposed in the literature regarding this technique.

5.1 Baldasso's Hypothesis and Its Refutation

Baldasso (2008) hypothesizes that Ratdolt created his figures by bending and cutting thin metal strips, probably made of zinc or copper, and then inserting them into a base of terracotta or not-yet-cooled glass, which had been previously engraved with guiding grooves for the design of the figures (fig. 9). The underlying idea is that, instead of engraving figures into wooden blocks, Ratdolt might have used flexible metal strips mounted on a rigid support to obtain more precise forms. This hypothesis appears to be supported by Mayor (1971), who, referring to Ratdolt's Editio Princeps, describes geometric figures made with bent metal strips fixed into a base of plaster or lead (Mayor, 1971, as cited in Baldasso, 2008). Sachiko Kusukawa (2000) also supports Baldasso's hypothesis, emphasizing that in the early years of movable type printing, it was particularly difficult to achieve fine lines using woodcut techniques.

Figures 6 and 7 illustrate the difference in representation techniques between Ratdolt's diagrams and later ones produced using the woodcut technique. Fig. 6 depicts the dodecahedron in the Editio Princeps of Elements, printed by Ratdolt in 1482, while Fig. 7 shows the dodecahedron in Euclide Megarense by Niccolò Tartaglia, printed by Venturino Ruffinelli in 1543.

However, Baldasso's hypothesis presents several issues. A terracotta or glass base would likely not withstand the pressure of the printing press. Moreover, the technique of fitting metal strips together would not allow for the creation of tangents, which are present in Ratdolt's edition. For instance, when producing two tangent circles, two circular metal strips would need to be used, which would result in two visible lines in the print, something that does not occur in Ratdolt's edition.

5.2 Our Hypotheses: Hypothesis 1

Given the mechanical limitations of Baldasso's proposal — particularly the fragility of the base and the difficulty in achieving tangency between curved elements — we believe a more robust and reproducible method would have involved a casting process. We proposed several solutions regarding Ratdolt's secret technique. We hypothesize it is likely that a casting process was involved. In a mold made of clay or another heat-resistant material, metal strips, previously shaped and cut, were inserted and then removed, creating grooves that formed a negative of the diagram (fig. 11). Molten metal was then poured into these grooves, producing a positive form of the diagram. With this technique, it would also have been possible to insert movable type into the mold, allowing both the diagrams and labels to be cast together. This approach would have allowed for greater integration between text and figure, making page composition more efficient. However, it is also likely that Ratdolt used two passes in printing, applying masks to the diagrams in areas where the labels were to be printed.

Initially, we hypothesized that the mold could have been made of sand. We initially considered the use of sand, due to its malleability in creating detailed molds. However, Khan, Sheikh, and Al-Shaer (2017), as well as Altan Turkeli (2016), note that sand began to be used as a casting material around the 16th century.

To achieve a perfect final shape, it is important to properly design the negative, making it essential to accurately shape the metal strips to be inserted into the mold. This task could be accomplished using two techniques: turning and rolling.

Turning allows for the creation of cylinders, which can then be shaped into circular metal strips as described in paragraph 5.4. Chondros (2021), discussing Giovanni Fontana (1395–1455) and his Bellicorum instrumentorum liber, notes the presence of a hand-crank lathe in the illustrations. It is likely that wood was turned during Ratdolt's time; indeed, Father Plumier in 1701 referenced the challenges of turning iron pieces in his book L'Art de Tourner, noting that hand-crank lathes with foot pedals became widespread only in the early 18th century (López De Lacalle & Lamikiz, 2009).

The metalworking technique allowing to produce the iron strip, rolling, was known to goldsmiths, and hand-operated rollers were in use as early as the 14th century. However, the modern concept of a rolling mill likely originates from a design by Leonardo da Vinci (Ray, 2016).

5.3 Our Hypotheses: Hypothesis 2

The second hypothesis suggests that Ratdolt may have created the geometric diagrams, including the labels, using a technique similar to that used for producing movable type. This hypothesis may be plausible because it would simplify composition and printing management, as the material would be the same as that used for movable type and would offer similar durability and resistance—although it would also result in very heavy blocks that are difficult to cool.

Movable type was made by first engraving a punch, a carbon-steel bar with a letter incised at one end using files and other specialized tools. The letter was engraved in reverse. Once the punch was prepared, it was used to create a matrix, typically made of copper. The punch was struck onto the matrix, leaving an impression of the letter in reverse. The matrix was then placed into a mold, and molten metal (usually lead and antimony) was poured in to create the movable type. The final product was a small block with the letter on top, in reverse, just like the punch. The type was then finished to reach the correct height for printing (Chappell & Bringhurst, 1999).

Given this technique, Ratdolt may have created the geometric diagrams similarly. After shaping the metal strips to match the lines of the diagram, he could have struck them onto a copper matrix, along with the punches for the labels. In this case, the metal strips would have acted as "punches" for the diagrams (fig. 11). He could then have inserted the matrix into a custom mold and poured molten metal into it, creating a metal form that included both the diagram and the letters. One drawback of this hypothesis is that the metal strips, even when tempered, are relatively fragile and difficult to drive into the copper matrix without bending or deforming them. This would have made the preparation of precise matrices particularly challenging.

Unfortunately, there is not enough evidence to confirm either of these hypotheses. In his letter to Doge Mocenigo, Ratdolt writes: "Therefore, since this alone was hindering the common benefit that everyone derives from these things, through my own effort—not without great labor—I managed to ensure that, with the same ease with which the types are impressed, geometric figures too may be produced"[5] (Ratdolt, 1482, as translated by the author)

This statement suggests that he used a technique similar to that of movable type for the diagrams. "Imprimuntur" may be interpreted both as "printed" and "impressed" (as in a matrix), which suggests that both Hypothesis 2 and 1 may be meaningful. However, without further evidence, these hypotheses remain speculative. Further testing through casting and print trials is necessary to verify these claims. It is also possible that Ratdolt used woodcut techniques to produce the diagrams, and this possibility should also be tested through casting and printing experiments.

5.4 Reconstructing the process of the circular metal strips

One of the authors personally undertook an experimental reconstruction (fig. 8) by turning a boxwood cylinder, producing a cylinder with varying diameters. Woodturning (and metalturning) was already practiced at the time of Ratdolt. For instance, Jacques Lefèvre d'Étaples described the construction of a wooden sphere using a lathe in the 1495 printed edition of his commentary on *De Sphaera* by Sacrobosco, published by Johannes Higman for Wolfgangum Hopyl. The edition even includes an illustration of a semicircular blade designed for that purpose (Oosterhoff, 2020).

They also attempted to hammer a 1.8 mm iron strip into a circular shape, successfully forming a ring. The resulting circles were then tempered—through separate attempts using oil, water, and urine—and subsequently fixed with wax onto a metal base, to be used either in pressing a copper plate matrix or possibly in the printing process.

The impression on the copper plate revealed several issues due to the width of the glyph, which made it difficult to apply uniform pressure with a hammer. This may suggest that a screw press might have been employed instead. Future experiments will involve reducing the height of the metal strip and casting lead and antimony into the copper matrix.

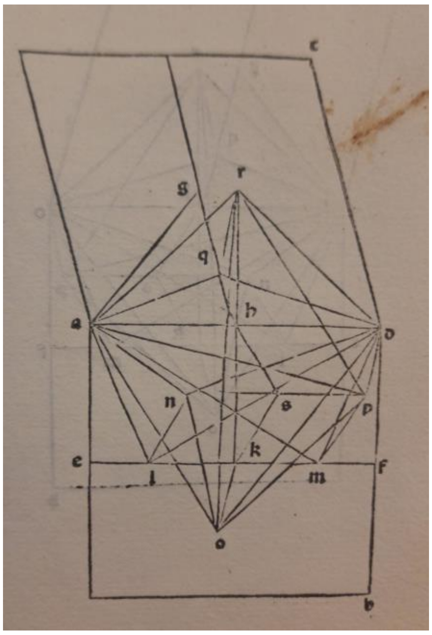

Fig. 6: dodecahedron from Ratdolt's *Preclarissimus liber elementorum Euclidis,* printed in 1482. The images are under Public Domain Mark 1.0.

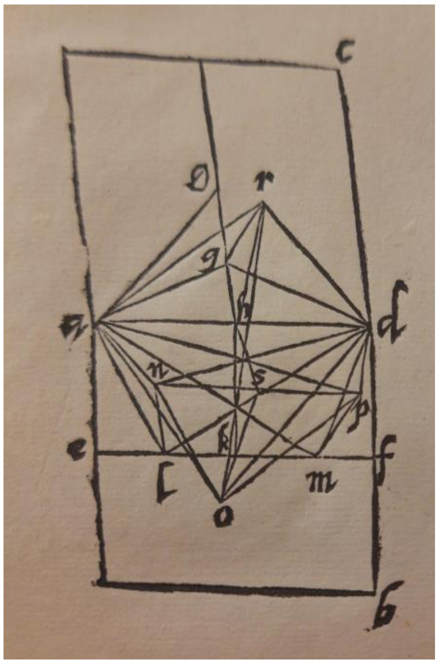

Fig. 7: dodecahedron from Niccolò Tartaglia's *Euclide megarense philosopho*, printed in 1543. The images are under Public Domain Mark 1.0.

[5] "Itaque cum hoc ipsum tantummodo communi omnium utilitati quae ex his percipitur, obstaret, mea industria non sine maximo labore effeci, ut qua facilitate litterarum elementa imprimuntur, ea etiam geometricae figurae conficerentur."

6. *Conclusions*

This article has examined the graphic representations of Euclid's dodecahedron, tracing their historical and visual development from early manuscript traditions through influential printed editions. The first chapter explored how Euclidean geometry, specifically the dodecahedron described in Proposition XVII of Book XIII, was visually represented across various translations and editions. It highlighted the complex interplay between mathematical abstraction and empirical representation, demonstrating how different periods and translators visually interpreted Euclidean geometry, influencing the ways in which geometric knowledge was transmitted.

The analysis further focused on graphic representations of Euclid's dodecahedron, from Erhard Ratdolt's influential 1482 edition to the developments of the 19th century, reveals a slow but significant evolution in the visualization of geometric forms within typography, driven by technical innovations, visual strategies, and methodological constraints.

In contrast to hand-drawn versions, early printed representations maintained uniformity in line thickness, complicating the establishment of visual hierarchies among geometric elements. Although Ratdolt's edition was innovative and remarkable for the quality of its diagrams, it did not yet clearly implement variations in line thickness, despite explicitly recognizing the educational importance of visual aids for understanding geometry. It was only in the 19th century that explicit visual hierarchies using graphic variations were introduced.

The article's discussion emphasizes the interplay between theoretical and practical considerations in representing geometric forms, demonstrating how pedagogical intentions—namely, the underlying theories about how and why geometry should be taught— influenced the development of didactic strategies and the graphic solutions. Ratdolt's explicit recognition of visual aids as essential for mathematical comprehension highlights a shift toward more didactically focused visual strategies, aiming to improve the effectiveness of geometric illustrations. This evolution reflects broader cultural and intellectual shifts, including the Renaissance emphasis on perspective and visual perception.

Furthermore, the article explores potential techniques employed by Ratdolt for printing geometric figures, suggesting hypotheses based on casting or stamping methods analogous to those used in movable type printing. However, due to the

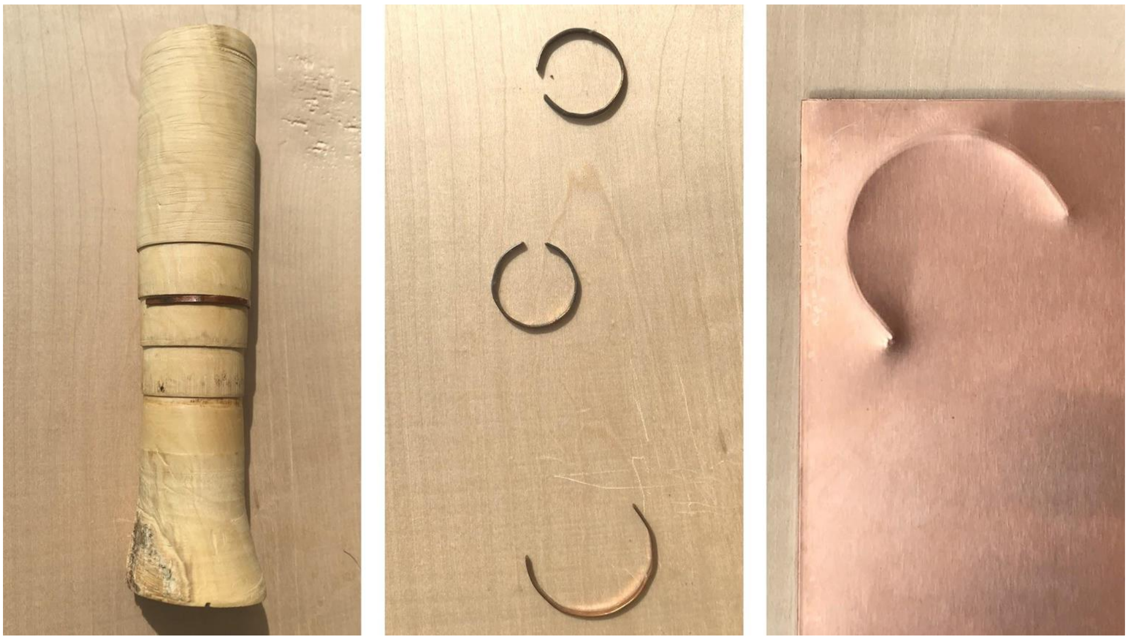

Fig. 8: This is an example of the template made with the lathe, of circles obtained by bending iron rods, and of a matrix produced by hammering the iron in a copper plate.

absence of definitive evidence, these remain hypotheses requiring further experimental verification in the development of this research.

Overall, the study highlights how the graphical evolution of Euclidean diagrams reflects a constant tension between mathematical abstraction, didactic necessities, and technical progress, underscoring the significance of graphic representations not merely as visual aids, but as intrinsic and essential components in the transmission of mathematical knowledge.

Limitations of this study include the scarcity of direct historical documentation and tangible evidence concerning the specific techniques employed by Ratdolt and his contemporaries. Future research will focus on practical experiments and reconstructions of these hypothesized printing methods, alongside a more comprehensive analysis of later editions and manuscripts, to deepen understanding of the historical development of geometric representations.

7. *Acknowledgements*

This work is the result of a collective reflection developed through ongoing dialogue, shared research, and interdisciplinary collaboration. The themes explored in the article have emerged from joint discussions, and the overall structure reflects a shared intellectual path.

While the article represents a unified project, we wish to acknowledge that Chapter 1 was written by Massimiliano Ciammaichella, Chapter 3 and 5 was written by Chiara Monaco, and Chapter 2, 4 and paragraph 5.3 was authored by Luciano Perondi. Abstract and Conclusions are cured by authors.

All research documentation related to this study is available on the Open Science Framework (OSF) under the project page: [Geometry, Pedagogy, and Printing Techniques: An Analysis of Euclid's Dodecahedron] – [https://osf.io/hpgka/?view_only=7c1fd21e88f54a6aa137d2f56fa9d7d5].

We would like to thank all those who, directly or indirectly, supported the development of this work through feedback, bibliographic suggestions, and participation in the seminars that nourished this reflection.

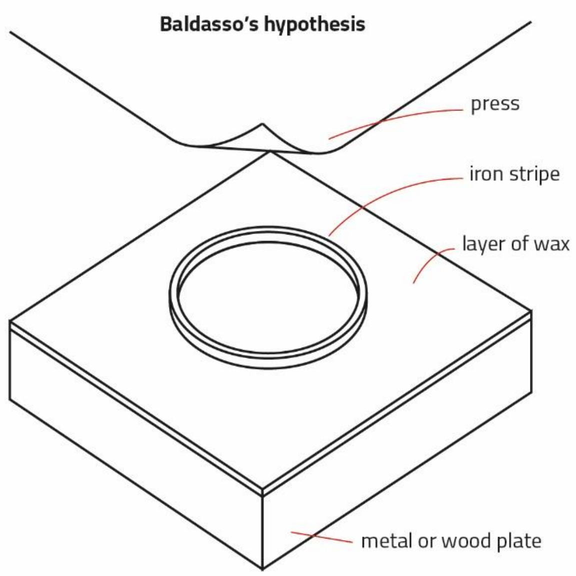

Fig. 9: visual reconstruction concerning the Baldasso's hypothesis.

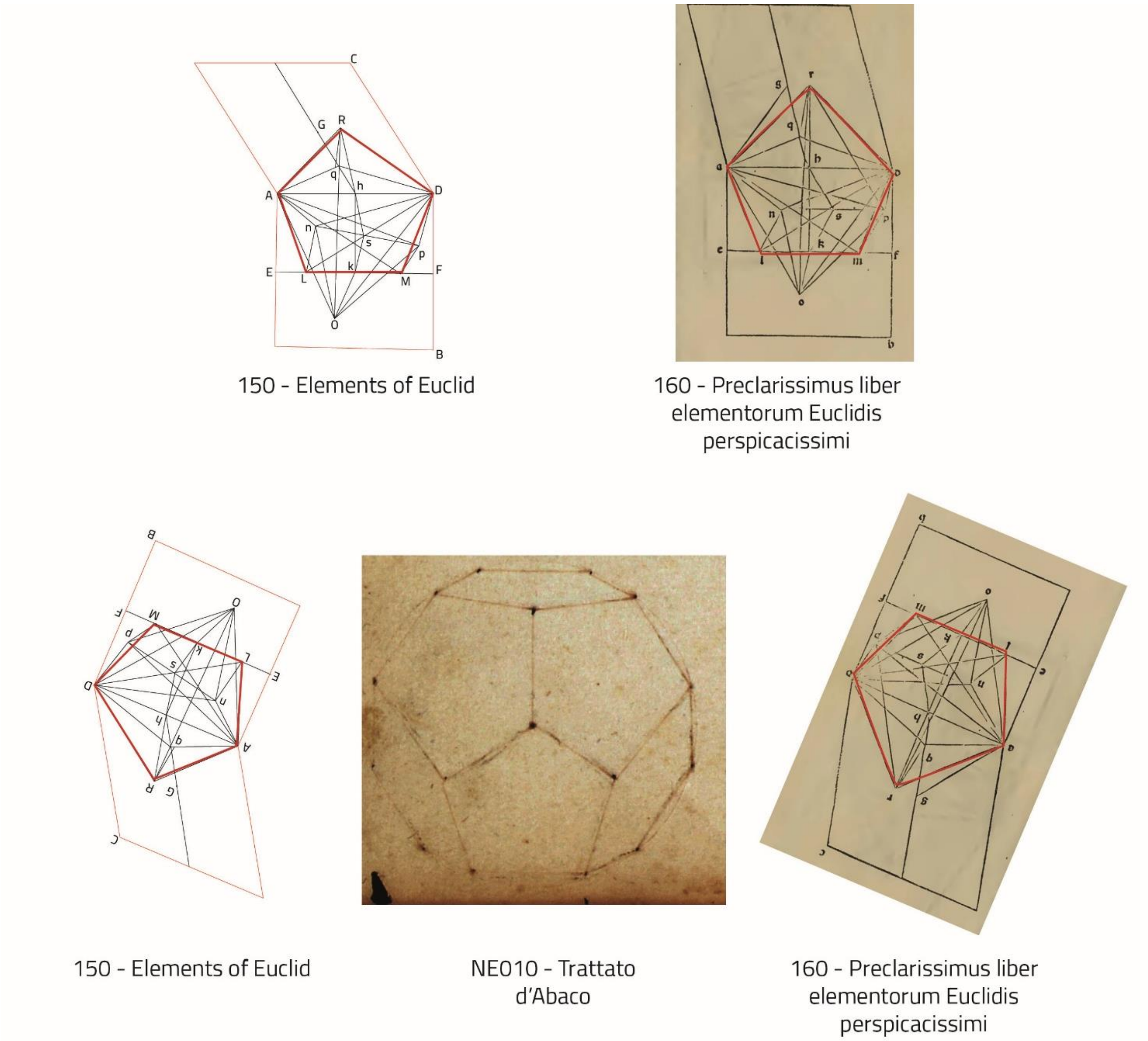

Fig. 10: Comparison of dodecahedrons in 160 - Preclarissimus liber elementorum Euclidis perspicacissimi (Venice, 1482, by Erhard Ratdolt). Held at the Boston Public Library, Rare Books Department. Public Domain Mark 1.0, 150 - Elements of Euclid (Vat.lat.2224, 1457, redrawn by the authors), and NE010 - Trattato d'Abaco (ms. Ashb. 359*, c. 110v). Held at the Biblioteca Medicea Laurenziana, Florence. Reproduced by the authors with permission of the Italian Ministry of Culture (MiC). All further reproduction by any means is prohibited.

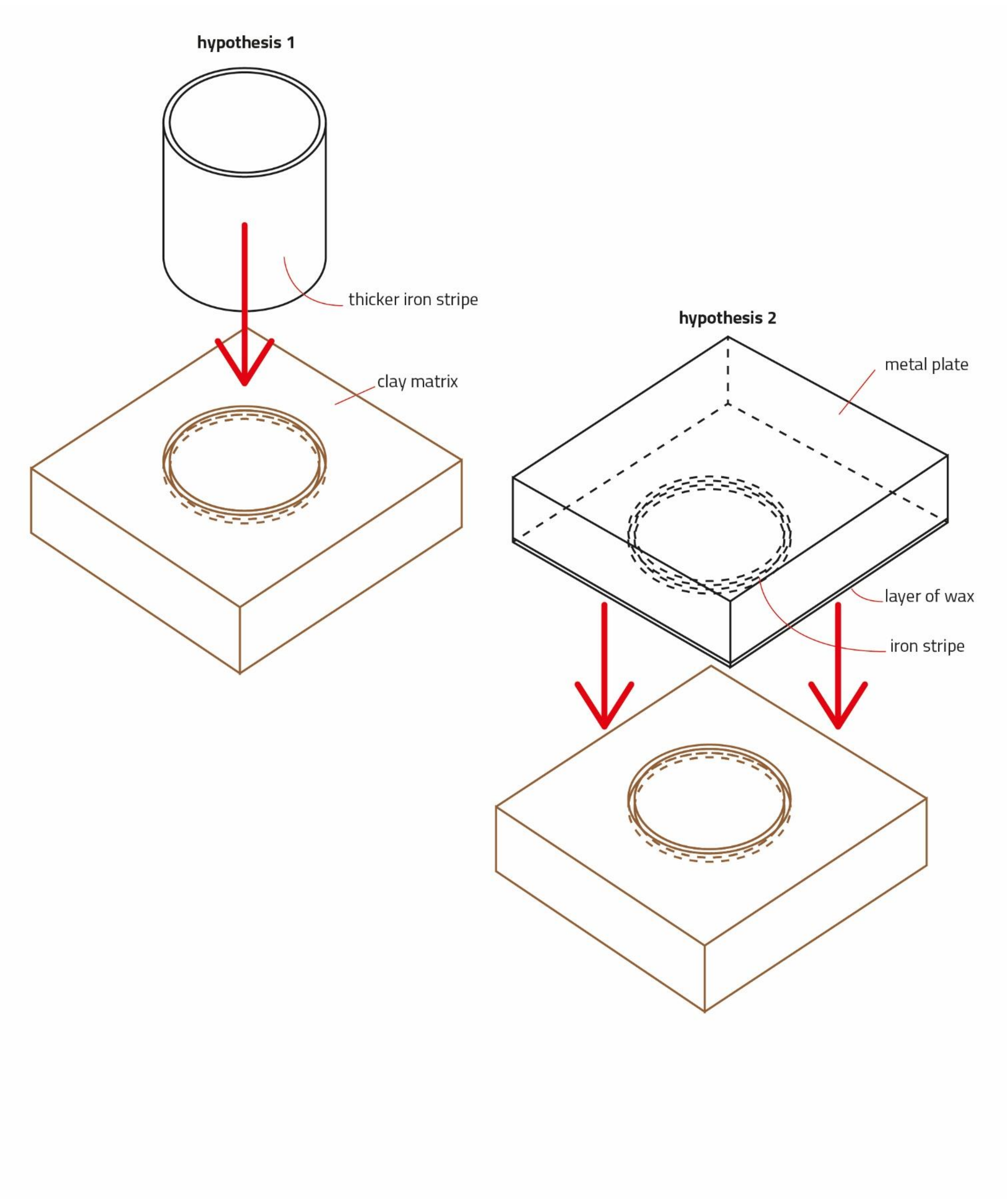

Fig. 11: Visual reconstruction of two experimental hypotheses concerning the production technique of Ratdolt's geometric diagrams.

References

Baldasso, R. (2008). La stampa dell'*editio princeps* degli *Elementi* di Euclide (Venezia, Erhard Ratdolt, 1482). In L. Pon & C. Kallendorf (Eds.), *The Books of Venice* (pp. 61–100). New Castle, DE, and Venice, Italy: Oak Knoll Press, Biblioteca Nazionale Marciana, and La Musa Talìa.

Banker, J. R. (2003). *The culture of San Sepolcro during the youth of Piero della Francesca*. Ann Arbor, MI: University of Michigan Press.

Bernante, A. (2020). *L'Euclide di Bartolomeo Zamberti e il Rinascimento della matematica e delle arti a Venezia tra la fine del Quattrocento e l'inizio del Cinquecento* (Tesi di laurea magistrale). Università Ca' Foscari Venezia.

Busard, H. L. L. (1984). *The Latin Translation of the Arabic Version of Euclid's Elements Commonly Ascribed to Gerard of Cremona Introduction, edition and critical apparatus*. Leiden: E. J. Brill.

Byrne, O. (1847). *The first six books of the Elements of Euclid: In which coloured diagrams and symbols are used instead of letters for the greater ease of learners*. London, UK: William Pickering.

Ciocci, A. (2009). *Luca Pacioli tra Piero della Francesca e Leonardo*. Sansepolcro, Italia: Aboca.

Ciocci, A. (2020). Luca Pacioli, Leonardo da Vinci e il disegno dei poliedri. In M. Martelli (Ed.), *Arte e matematica in Luca Pacioli e Leonardo da Vinci* (pp. 43–86). Città di Castello: Edizioni Nuova Prhomos.

Commandino, F. (1572). *Euclidis Elementorum libri 15. Unà cum scholijs antiquis*. Pesaro, Italia: Camillo Franceschini.

De Young, G. (2004). The Latin translation of Euclid's *Elements* attributed to Gerard of Cremona in relation to the Arabic transmission. *Suhayl. Journal for the History of the Exact and Natural Sciences in Islamic Civilisation*, *4*, 311–328.

Dürer, A. (1525). *Underweysung der Messung mit dem Zirckel und Richtscheyt in Linien, Ebnen unnd gantzen Corporen*. Nürnberg, Germania: Hieronymus Andreae.

Fazzini, G. (2003). Luca Pacioli e Venezia. In M. Emmer (Ed.), *Matematica e cultura 2003* (pp. 153–158). Milano, Italia: Springer-Verlag Italia.

Gavagna, V. (2009). La tradizione euclidea nel Rinascimento. In F. Commandino (Ed.), *De gli Elementi d'Euclide volgarizzati da Federico Commandino* (pp. 1–10). Urbino, Italia: Accademia Raffaello. Retrieved from https://www.researchgate.net/publication/296195394

Joost-Gaugier, C. L. (2008). *Pitagora e il suo influsso sul pensiero e sull'arte*. Roma, Italia: Arkeios.

Kemp, M. (1990). *The science of art: Optical themes in Western art from Brunelleschi to Seurat*. New Haven, CT: Yale University Press.

Knorr, W. (1989). "La Croix des Mathématiciens": The Euclidean theory of irrational lines. *Bulletin of the American Mathematical Society*, *9*(1), 41–69.

Lee, E. (2018). Let the diagram speak: Compass arcs and visual auxiliaries in printed diagrams of *Euclid's Elements*. *Endeavour*, *42*(2–3), 78–98.

Menge, H., & Heiberg, J. L. (1885). *Euclidis Opera Omnia*. Lipsia, Germania: In Aedibus B. G. Teubneri.

Nenci, E. (1998). *Bernardino Baldi. Le vite de' matematici: Edizione annotata e commentata della parte medievale e rinascimentale*. Milano, Italia: Franco Angeli.

Oosterhoff, R. J. (2020). A lathe and the material sphaera: Astronomical technique at the origins of the cosmographical handbook. In M. Valleriani (Ed.), *De sphaera of Johannes de Sacrobosco in the early modern period: The authors of the commentaries* (pp. 25–52). Cham, Switzerland: Springer. http://dx.doi.org/10.1007/978-3-030-30833-9_2

Pacioli, L. (1509). *Divina proportione*. Venezia, Italia: Paganino de' Paganini.

Pagliara, P. N. (1997). Francesco di Benedetto Cereo da Borgo San Sepolcro (Francesco del, di Borgo; *Franciscus Burgensis*). In *Dizionario biografico degli italiani* (Vol. 49, pp. 692–696). Roma, Italia: Istituto dell'Enciclopedia Italiana.

Pergola, R. (2009). *Ex arabico in latinum: Traduzioni scientifiche e traduttori nell'Occidente medievale*. *Studi di Glottodidattica*, *3*(3), 74–105.

Riccardi, P. (1887). *Saggio di una bibliografia euclidea*. Bologna: Tipografia Gamberini e Parmeggiani.

Scolari, M. (2006). *Il disegno obliquo: Una storia dell'antiprospettiva*. Venezia, Italia: Marsilio.

Vasari, G. (1550). *Le vite de' più eccellenti architetti, pittori, et scultori italiani, da Cimabue insino a' tempi nostri: Descritte in lingua toscana, da Giorgio Vasari pittore aretino. Con una sua utile & necessaria introduzzione a le arti loro*. Firenze, Italia: Lorenzo Torrentino.

Zamberti, B. (1505). *Euclidis Megarensis philosophi platonici mathematicarum disciplinarum janitoris*. Venezia, Italia: Giovanni Tacuino.